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The confluent algorithm in second-order supersymmetric quantum mechanics

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Abstract
The confluent algorithm, a degenerate case of the second-order supersymmetric quantum mechanics, is studied. It is shown that the transformation function must asymptotically vanish to induce non-singular final potentials. The technique can be used to create a single level above the initial ground state energy. The method is applied to the free particle, one-soliton well and harmonic oscillator.

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1. Introduction

The second-order supersymmetric quantum mechanics (2-SUSY QM), which involves second-order differential intertwining operators [1–8], has proved useful to surpass the difficulty of 'modifying' the excited state levels inherent to the standard first-order 1-SUSY QM. In fact, in the 2-SUSY treatment we do not respect the restrictions imposed by 1-SUSY on the transformation functions $u_1(x), u_2(x)$, i.e., they can have nodes but induce a non-singular 2-SUSY transformation [9–12]. In this way, potentials with two extra bound states above the ground state energy of the initial Hamiltonian H (or above the lowest band edge if $V(x)$ is periodic) have been recently generated [9, 11, 12]. A different atypical method employing two complex conjugate factorization energies has been implemented as well generating in this case families of real isospectral 2-SUSY partner potentials of $V(x)$ [13].

There is yet another situation worth studying in detail, namely, when the two factorization energies tend to a common *real* value ϵ. This so-called confluent algorithm was used to generate a particular family of isospectral oscillator potentials [14]. However, we have not detected a generic analysis (for arbitrary factorization energies ϵ) of the properties of the transformation function $u(x)$ ensuring that the final potential will be non-singular. This is the subject of the present paper, which has been organized as follows. In section 2, an alternative view of the confluent 2-SUSY algorithm will be elaborated upon. The restrictions imposed onto $u(x)$

in order to obtain non-singular final potentials will be analysed in section 3. In section 4, we will apply the technique to the free particle, one-soliton well and standard harmonic oscillator.

2. Second-order supersymmetric quantum mechanics

The second-order supersymmetric quantum mechanics (2-SUSY QM) is a particular realization of the standard supersymmetry algebra with two generators [1–6]:

$$\{Q_j, Q_k\} = \delta_{jk} H_{\rm ss} \qquad [H_{\rm ss}, Q_j] = 0 \qquad j, k = 1, 2 \tag{1}$$

where $Q_1 = (Q^\dagger + Q)/\sqrt{2}$, $Q_2 = (Q^\dagger - Q)/(\mathrm{i}\sqrt{2})$,

$$Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \qquad Q^\dagger = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix} \tag{2}$$

$$H_{\rm ss} = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} = \begin{pmatrix} (\widetilde{H} - \epsilon_1)(\widetilde{H} - \epsilon_2) & 0 \\ 0 & (H - \epsilon_1)(H - \epsilon_2) \end{pmatrix} \tag{3}$$

and H, $\widetilde{H}$ are two intertwined Schrödinger Hamiltonians:

$$\widetilde{H} A = AH \tag{4}$$

$$\widetilde{H} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \widetilde{V}(x) \qquad H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) \tag{5}$$

$$A = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \eta(x)\frac{\mathrm{d}}{\mathrm{d}x} + \gamma(x). \tag{6}$$

The relations between $\eta(x)$, $\gamma(x)$, $V(x)$ and $\widetilde{V}(x)$, are

$$\widetilde{V} = V + 2\eta' \tag{7}$$

$$\gamma = d - V + \eta^2/2 - \eta'/2 \tag{8}$$

$$\eta\eta'' - (\eta')^2/2 + \eta^2(\eta^2/2 - 2\eta' - 2V + 2d) + 2c = 0 \tag{9}$$

where, in terms of $c \in \mathbb{R}$, $d \in \mathbb{R}$, the factorization energies read $\epsilon_1 = d + \sqrt{c}$, $\epsilon_2 = d - \sqrt{c}$. Suppose that $V(x)$ is a solvable potential given exactly, then $\widetilde{V}(x)$ will be effectively determined if we find explicit solutions $\eta(x)$ to the nonlinear second-order differential equation (9). Depending on the sign of c, two essentially different cases arise.

If $c \neq 0$ then $\epsilon_1 \neq \epsilon_2$, and we must look for solutions of the two Riccati equations:

$$\beta_i' + \beta_i^2 = V - \epsilon_i \qquad i = 1, 2. \tag{10}$$

Having $\beta_1(x)$, $\beta_2(x)$, we get two different equations for $\eta(x)$ (see, e.g., [6, 13])

$$\eta' = \eta^2 + 2\beta_1\eta + \epsilon_2 - \epsilon_1 \tag{11}$$

$$\eta' = \eta^2 + 2\beta_2\eta + \epsilon_1 - \epsilon_2. \tag{12}$$

By subtracting them, we arrive at a finite difference algorithm for $\eta(x)$

$$\eta(x) = (\epsilon_1 - \epsilon_2)/(\beta_1 - \beta_2). \tag{13}$$

On the other hand, the *confluent* case arises for $c = 0$ implying that $\epsilon_1 = \epsilon_2 \equiv \epsilon = d$. In this situation, we look for solutions to just one Riccati equation [14]

$$\beta' + \beta^2 = V - \epsilon \tag{14}$$

and $\eta(x)$ satisfies an equation arising when $\epsilon_1 = \epsilon_2$ in (11) and (12):

$$\eta' = \eta^2 + 2\beta\eta. \tag{15}$$

This is the Bernoulli equation, whose general solution is given by

$$\eta(x) = -w'(x)/w(x) \tag{16}$$

where

$$w(x) = w_0 - \int e^{2\int\beta(x)\,dx}\,dx. \tag{17}$$

In the next section we will analyse the conditions which grant that the confluent 2-SUSY transformations are non-singular. This provides the simplest way of ensuring that, departing from an exactly solvable initial potential $V(x)$, we arrive at an exactly solvable regular $\widetilde{V}(x)$ as well (see, e.g., the discussion in [15]).

3. Confluent non-singular transformations

Let us first express the confluent formulae of section 2 in terms of solutions of the initial Schrödinger equation obtained from (14) by the change $\beta(x) = u'(x)/u(x)$:

$$-u''(x) + V(x)u(x) = \epsilon u(x). \tag{18}$$

Thus, up to an unimportant constant factor (see (16)), the key function $w(x)$ becomes

$$w(x) = w_0 - \int_{x_0}^{x} u^2(y)\,dy \tag{19}$$

and the confluent 2-SUSY potential $\widetilde{V}(x)$ is given by

$$\widetilde{V}(x) = V(x) - 2[w'(x)/w(x)]'. \tag{20}$$

It is clear now that in order to arrive at real non-singular potentials $\widetilde{V}(x)$ we have to use real solutions $u(x)$ of (18) inducing a nodeless $w(x)$. Let us note that

$$w'(x) = -u^2(x) \tag{21}$$

meaning that $w(x)$ is decreasing monotonically, so the simplest way of avoiding its zeros is to look for the appropriate asymptotic behaviour for $u(x)$. Two different situations are worth considering.

(i) Suppose first that $\epsilon = E_m$ is one of the discrete eigenvalues of H and the transformation function is the corresponding *normalized* physical eigenfunction, $u(x) = \psi_m(x)$. Denote by ν_+ the following finite integral:

$$\nu_+ \equiv \int_{x_0}^{\infty} u^2(y)\,dy. \tag{22}$$

It is straightforward to show that

$$\lim_{x\to-\infty} w(x) = w_0 - \nu_+ + 1 \equiv \nu + 1 \tag{23}$$

and

$$\lim_{x\to+\infty} w(x) = \nu. \tag{24}$$

It turns out that $w(x)$ is nodeless if either both limits are positive or both negative, leading to the ν-domain where the confluent 2-SUSY transformation is non-singular:

$$\nu \in \mathbb{R}\backslash(-1, 0) = (-\infty, -1] \cup [0, \infty). \tag{25}$$

(ii) Suppose now that the transformation function $u(x)$ is a *non-normalizable* solution of (18) associated with a real factorization energy $\epsilon \notin \mathrm{Sp}(H)$ such that

$$\lim_{x\to\infty} u(x) = 0 \qquad \text{and} \quad \nu_+ \equiv \int_{x_0}^{\infty} u^2(y)\,\mathrm{d}y < \infty. \tag{26}$$

If this is the case we can show that

$$\lim_{x\to-\infty} w(x) = w_0 + \int_{-\infty}^{x_0} u^2(y)\,\mathrm{d}y = \infty \tag{27}$$

and

$$\lim_{x\to+\infty} w(x) = w_0 - \nu_+ \equiv \nu. \tag{28}$$

By comparing both limits and taking into account that $w(x)$ is decreasing monotonically, it turns out that $w(x)$ is nodeless if

$$\nu \geqslant 0. \tag{29}$$

Let us note that the same ν-restriction holds in the case when

$$\lim_{x\to-\infty} u(x) = 0 \qquad \text{and} \quad \nu_- \equiv \int_{-\infty}^{x_0} u^2(y)\,\mathrm{d}y < \infty \tag{30}$$

though now $\nu \equiv -(w_0 + \nu_-)$.

Once the regularity of the confluent 2-SUSY algorithm is assured, we analyse the spectrum of $\widetilde{H}$. From the intertwining relationship (4) and the factorizations in (3) we immediately obtain normalized eigenstates $|\widetilde{\psi}_n\rangle$ of $\widetilde{H}$ provided that ν satisfies either (25) in case (i) or (29) in case (ii) and $A|\psi_n\rangle \neq 0$:

$$|\widetilde{\psi}_n\rangle = (E_n - \epsilon)^{-1} A|\psi_n\rangle \tag{31}$$

(so in case (i) we cannot obtain $|\widetilde{\psi}_m\rangle$ of (31) because $A|\psi_m\rangle = 0$). The orthonormal set $\{|\widetilde{\psi}_n\rangle, n = 0, 1, 2, \ldots\}$ so constructed is not automatically complete (we have yet to analyse the existence or not of an extra normalizable function $\widetilde{\psi}$ belonging to the Kernel of $A^\dagger$ which is orthogonal to all the $|\widetilde{\psi}_n\rangle, n = 0, 1, 2, \ldots$). To find $\widetilde{\psi}$ explicitly, let us factorize $A^\dagger$ as follows:

$$A^\dagger = \left[\frac{\mathrm{d}}{\mathrm{d}x} + \beta(x)\right]\left[\frac{\mathrm{d}}{\mathrm{d}x} - \beta(x) - \eta(x)\right]. \tag{32}$$

It turns out that the $\widetilde{\psi} \in \mathrm{Ker}(A^\dagger)$ we are looking for is annihilated by the second factor operator of (32)

$$\widetilde{\psi}(x) = n_0\,\mathrm{e}^{\int[\beta(x)+\eta(x)]\,\mathrm{d}x} = n_0 u(x)/w(x) \tag{33}$$

where n_0 is a constant. It is straightforward to check that $\widetilde{\psi}(x)$ is a normalized eigenfunction of $\widetilde{H}$ with eigenvalue ϵ for $\nu \in \mathbb{R}\backslash[-1, 0]$ in case (i) with $n_0 = \sqrt{\nu(\nu+1)}$ and for $\nu > 0$ in case (ii) with $n_0 = \sqrt{\nu}$. On the other hand, $\widetilde{\psi}(x)$ becomes non-normalizable for $\nu = -1, 0$ in case (i) or for $\nu = 0$ in case (ii). Thus, when $\epsilon = E_m$ and $u(x) = \psi_m(x)$ it turns out that $\mathrm{Sp}(\widetilde{H}) = \mathrm{Sp}(H)$ if $\nu \in \mathbb{R}\backslash[-1, 0]$ while the level E_m is not present in $\mathrm{Sp}(\widetilde{H})$ for $\nu = -1, 0$ (in this case E_m has been 'deleted' in order to generate $\widetilde{H}$). On the other hand, when $\epsilon \notin \mathrm{Sp}(H)$ and $u(x)$ obeys either (26) or (30) it turns out that $\mathrm{Sp}(\widetilde{H}) = \{\epsilon\} \cup \mathrm{Sp}(H)$ for $\nu > 0$ but $\mathrm{Sp}(\widetilde{H}) = \mathrm{Sp}(H)$ for $\nu = 0$. For all the other ν-values ($\nu \in (-1, 0)$ in case (i) and $\nu < 0$ in case (ii)) it gives rise to a singularity in $\widetilde{V}(x)$ due to the existence of a zero in $w(x)$. We note, in particular, that case (ii) allows us to generate a single level above the ground state energy of H, a mechanism which cannot be directly implemented in the 1-SUSY treatment.

Let us remark that our confluent 2-SUSY procedure coincides with the Abraham–Moses generation technique of creating, deleting or changing the normalization of a single energy level [16] (see also [17, 18]). The same procedure, known as binary Darboux transformations [19], has been employed to generate bound states embedded in the continuum [20–22].

4. The simplest applications

Let us now analyse the simplest applications of the confluent 2-SUSY algorithm.

(a) Consider first the free particle for which $V(x) = 0$. For a fixed arbitrary $\epsilon < 0$ which does not belong to $\mathrm{Sp}(H)$ there are two asymptotically vanishing transformation functions:

$$u(x) = \sqrt{2k}\,\mathrm{e}^{\pm kx} \qquad \epsilon = -k^2. \tag{34}$$

A direct calculation leads to

$$w(x) = \mp 2\,\mathrm{e}^{\pm k(x+x_1)} \cosh[k(x-x_1)] \tag{35}$$

where $\nu = \mathrm{e}^{\pm 2kx_1} > 0$. By substituting these expressions into (20) we obtain the Pöschl–Teller potential in both cases

$$\widetilde{V}(x) = -2k^2 \operatorname{sech}^2[k(x-x_1)] \tag{36}$$

which has a bound state at $\epsilon = -k^2$.

(b) Now take the previous Pöschl–Teller as the initial potential:

$$V(x) = -2k_0^2 \operatorname{sech}^2(k_0 x) \tag{37}$$

and denote the ground state energy as usual, $E_0 = -k_0^2$.

Let us consider first the case when $\epsilon = E_0$ and $u(x)$ is the normalized ground state:

$$u(x) = \sqrt{\frac{k_0}{2}} \operatorname{sech}(k_0 x). \tag{38}$$

A straightforward calculation leads to

$$w(x) = \nu + \tfrac{1}{2} - \tfrac{1}{2}\tanh(k_0 x) \tag{39}$$

which produces once again the Pöschl–Teller potential:

$$\widetilde{V}(x) = -2k_0^2 \operatorname{sech}^2 k_0(x-x_1) \tag{40}$$

where $\tanh(k_0 x_1) = 1/(1+2\nu)$.

Suppose now that $\epsilon = -k^2 \neq E_0, k \in \mathbb{R}$. The solutions with the right asymptotic behaviour are here:

$$u(x) = \sqrt{2k}\,\mathrm{e}^{\pm kx}[k_0 \tanh(k_0 x) \mp k] \tag{41}$$

leading to

$$w(x) = \mp\{\nu + \mathrm{e}^{\pm 2kx}[k^2 + k_0^2 \mp 2kk_0 \tanh(k_0 x)]\}. \tag{42}$$

It turns out that the confluent 2-SUSY potential $\widetilde{V}(x)$ acquires the Bargmann form

$$\widetilde{V}(x) = -\frac{2(k_2^2 - k_1^2)[k_1^2 \operatorname{sech}^2 k_1(x-x_1) + k_2^2 \operatorname{csch}^2 k_2(x-x_2)]}{[k_1 \tanh k_1(x-x_1) - k_2 \coth k_2(x-x_2)]^2} \tag{43}$$

where for $k > k_0$ we need to take $k_1 = k_0, k_2 = k, \nu = (k^2 - k_0^2)\,\mathrm{e}^{\pm 2kx_2}, \mathrm{e}^{\pm 2k_0 x_1} = (k+k_0)/(k-k_0)$ while for $k < k_0$ we require $k_1 = k, k_2 = k_0, \nu = (k_0^2 - k^2)\,\mathrm{e}^{\pm 2kx_1}, \mathrm{e}^{\pm 2k_0 x_2} = (k+k_0)/(k_0-k)$.

(c) Finally, let us analyse the harmonic oscillator potential:

$$V(x) = x^2 \tag{44}$$

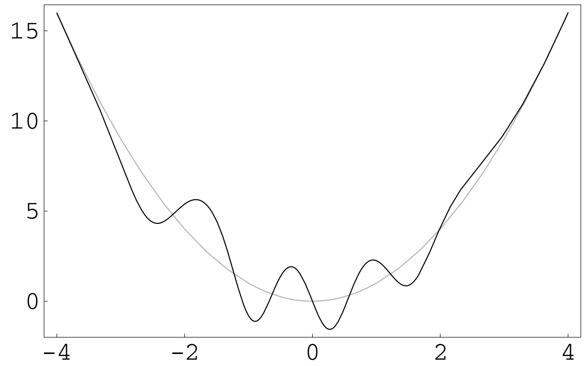

Figure 1. The confluent 2-SUSY partner potential (black curve) isospectral to the oscillator (grey curve) generated by employing the normalized eigenfunction (45) for $n = 3, \epsilon = E_3 = 7$ and $\nu = -5/4$.

which has a purely discrete spectrum composed of $E_n = 2n + 1, n = 0, 1, \ldots$ and eigenfunctions given by

$$\psi_n(x) = (\sqrt{\pi}2^n n!)^{-1/2}\, \mathrm{e}^{-x^2/2} H_n(x) \qquad n = 0, 1, \ldots \tag{45}$$

where $H_n(x)$ are the Hermite polynomials.

Let us suppose first that $\epsilon = E_m$ with m fixed, $u(x)$ being the corresponding normalized eigenfunction $\psi_m(x)$ of (45). The calculation of (19) with $x_0 = 0$ leads to

$$w(x) = \nu + \frac{1}{2} - x \sum_{s=0}^{m_0} \frac{(-1)^{m_0+s}\Gamma(m_1)(2x)^{2m_1-2s-1}}{2^{\delta+1}\sqrt{\pi}(m_1 - s)\Gamma\left(\delta + \frac{1}{2}\right)(m-2s)!s!} \\ \times {}_2F_2\left(m_1, m_1 - s; \delta + \frac{1}{2}, m_1 + 1 - s; -x^2\right) \tag{46}$$

where ${}_2F_2(a_1, a_2; b_1, b_2; z)$ is a generalized hypergeometric function [23], $m_0 = (m - \delta)/2, m_1 = (m + \delta + 1)/2, \delta = 0$ if m is even but $\delta = 1$ if m is odd. It turns out that $\widetilde{V}(x)$ is isospectral to the oscillator potential, a case illustrated in figure 1 for $\epsilon = 7$ $(m = 3)$ and $\nu = -5/4$. Let us note the already involved explicit expression of $w(x)$ in (46).

In turn, for $\epsilon \notin \mathrm{Sp}(H)$ the asymptotically vanishing Schrödinger solutions become

$$u(x) = \mathrm{e}^{-\frac{x^2}{2}}\left[{}_1F_1\left(\frac{1-\epsilon}{4}, \frac{1}{2}; x^2\right) \pm 2x\frac{\Gamma\left(\frac{3-\epsilon}{4}\right)}{\Gamma\left(\frac{1-\epsilon}{4}\right)}{}_1F_1\left(\frac{3-\epsilon}{4}, \frac{3}{2}; x^2\right)\right] \tag{47}$$

where ${}_1F_1(a, c; z)$ is the Kummer hypergeometric series. The explicit expression for $w(x)$ is too involved to be shown here (three infinite sums of kind (46) arise in this case). Alternatively, we performed a numeric calculation of $\widetilde{V}(x)$ for $\epsilon = 8$ taking the solution $u(x)$ of (47) with the upper plus sign and $w_0 = -5, x_0 = 0$ in (19) (see figure 2). The spectrum of $\widetilde{V}(x)$ is composed of the oscillator eigenenergies $E_n = 2n+1, n = 0, 1, \ldots$ plus a new level at $\epsilon = 8$. This illustrates clearly the possibility offered by the confluent 2-SUSY algorithm of creating one single level above the ground state energy of H.

We conclude that the second-order supersymmetric quantum mechanics is a powerful tool for designing in a simple way potentials with given spectra, a subject supplying us with solvable models with possible applications in the physical sciences (see e.g. [24]).

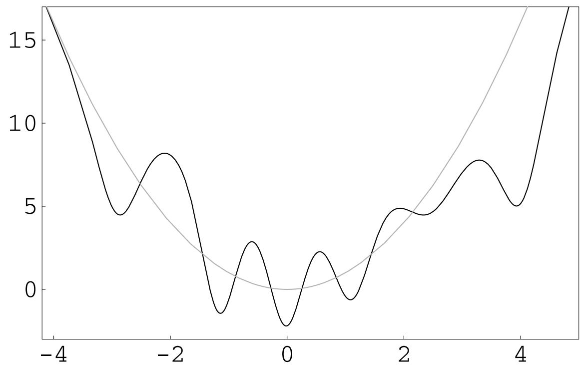

Figure 2. The confluent 2-SUSY partner potential $\widetilde{V}(x)$ (black curve) of the oscillator (grey curve) generated by employing the Schrödinger solution (47) with the upper + sign and $\epsilon = 8, w_0 = -5, x_0 = 0$. The potential $\widetilde{V}(x)$ has an extra bound state at $\epsilon = 8$ compared with the oscillator spectrum.

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